JOURNAL OF APPROXIMATION THEORY 5, 149–175 (1972)

Overdetermined Harmonic Polynomial Interpolation*

J. H. CURTISS

Department of Mathematics, University of Miami, Coral Gables, Florida 33146

Communicated by Oved Shisha

Received October 19, 1970

DEDICATED TO PROFESSOR J. L. WALSH ON THE OCCASION OF HIS 75TH BIRTHDAY

1. INTRODUCTION

Let C be a simple closed curve in the complex z-plane, $z = x + iy$, and let Int C denote the interior region of C. A function U from Int C to the reals is said to be harmonic on Int C if it is continuous on Int C, together with its first and second partial derivatives, and satisfies the Laplace equation

$$(\partial^2 U/\partial x^2) + (\partial^2 U/\partial y^2) = 0.$$

The classical Dirichlet problem for C and Int C is this: Given a continuous function u from C to the reals, find a function U, harmonic on Int C, such that at each point z_0 of C, $\lim_{z\to z_0} U(z) = u(z_0)$, $z \in \text{Int } C$.

This paper is concerned with a method for approximating the solution U on Int C in which a finite set Z_N of $2N + 1$ distinct points is chosen on C and harmonic polynomials $H_n(Z_N ; u; z)$ of respective degrees at most n, $n = 1, 2,..., N - 1$, are determined by best approximation to the boundary values in a least-squares sense on Z_N. The problem is to prove the existence of, and characterize sequences $\langle Z_N\rangle_{N=1}^{\infty}$ such that the corresponding double sequences $\langle H_n(Z_N ; u; z)\rangle$ converge to U on Int C.

A limiting case is that in which the least-squares approximation is perfect and the polynomial $H_n(Z_N ; u; z)$ is determined by direct interpolation to u at the nodes Z_N. In general it is necessary to take $n = N$ in that case. The problem of the convergence behavior of harmonic interpolation polynomials as the number of nodes increases was first proposed by Walsh [17]. It was solved by him for the case in which C is an ellipse [20], and it has recently been studied in some detail by the author [1, 3–5] for more general curves C.

There is empirical evidence that the behavior of the approximating

* This research was supported by National Science Foundation Grant GP-20049.

harmonic polynomials which are tied to the boundary values at nodes Z_N on C is improved when the polynomials are "overdetermined" in the sense that more nodes are used for a given degree n than would be required for simple interpolation. By improvement in behavior we mean chiefly that (1) for a given amount of computation, a closer approximation to the solution U is obtained when the nodes are placed on C in a manner which has proved successful for obtaining convergence in the direct interpolation case; and (2) convergence is less sensitive to the spacing of the nodes than it is with simple interpolation. Because of limitations of time the author has not attempted to supplement this paper with new numerical experiments to confirm (1).

However, support to (2) is given in Section 8 at the end of the paper, where a striking special case is presented in which C is the unit circle and the nodes are the roots of unity with a simple perturbation. For each continuous u, the sequence of overdetermined polynomials converges everywhere to the desired value on the open unit disk, but the sequence of direct interpolation polynomials is unbounded at an infinite set of points on the disk for an infinite set of continuous boundary-value functions u.

The existence of, and normal equations for calculating, the overdetermined least-squares polynomials $H_n(Z_N\,; u; z)$ are discussed in Section 2. In Section 3, the basic convergence theorem is announced (Theorem 3.1) and the problem of finding successful sequences $\langle Z_N \rangle$ on C is referred to the unit circle by conformal mapping. The Faber polynomials are thereafter taken to be the basis for the representation of $H_n(Z_N\,; u; z)$, and so play a central role in the convergence proof. Sections 4 and 5 are concerned with preliminaries (such as analytical requirements on the spacing of the points) for the convergence proof, which appears in Section 6. In Section 7, the behavior of $H_n(Z_N\,; u; z)$ is studied under the condition that n remains fixed, and $N \to \infty$. Section 8 contains the special case described above.

2. Existence and Structure of Overdetermined Harmonic Interpolation Polynomials

Let $Z_N = \{z_0, z_1, \ldots, z_{2N}\}$ be a set of $2N + 1$ (distinct) points on C. (It simplifies formulas slightly to use an odd number of nodes.) The criterion for fitting a harmonic polynomial $H(u; z)$ to $u(z)$ will be that

$$\sum_{k=0}^{2N} [H(u; z_k) - u(z_k)]^2$$

shall be a minimum. We examine this in terms of standard linear approximation theory. (See [7, Chap. 7].) Consider the linear space $\mathscr{L}_C$ of functions u from C to the reals, with restriction to Z_N, and with real scalars. Equip this space with an inner product $(u_1, u_2) = \sum_0^N u_1(z_k)\, u_2(z_k)$. The corresponding norm

$$\| u \| = \left\{ \sum_0^{2N} [u(z_k)]^2 \right\}^{1/2}$$

is such that the resulting normed linear space is strictly convex [7, p. 141]. Then if $v_1(z),..., v_m(z)$ are any m linearly independent elements of $\mathscr{L}_C$ (this means that the m vectors $\mathbf{v}_j = [v_j(z_0), v_j(z_1),..., v_j(z_{2N})]$, $j = 1,..., m$, are linearly independent), for each $u \in \mathscr{L}_C$ there exists a unique real linear combination $\sum_{j=1}^m C_j v_j(z)$ which minimizes $\| l - u \|$, where l runs through all real linear combinations of $v_1, v_2,..., v_m$ [7, pp. 137 ff].

A harmonic polynomial of degree at most n can be defined as the real part of a complex polynomial of degree n, and so is a real linear combination of the real-valued functions

$$1, z + \bar{z},..., z^n + \bar{z}^n, i(z - \bar{z}) \cdots i(z^n - \bar{z}^n), \tag{2.1}$$

where the bar denotes complex conjugate. Let, $1, p_1(z),..., p_n(z)$ be an arbitrary sequence of complex polynomials in which $p_j(z)$ is of degree exactly j, $j = 1,..., n$. With $g_0, a_1,..., a_n, b_1,..., b_n$ all real, the function

$$H(z) = g_0 + \sum_{j=1}^n a_j[p_j(z) + \overline{p_j(z)}] + \sum_{j=1}^n b_j i[p_j(z) - \overline{p_j(z)}]$$

$$= g_0 + \sum_{j=1}^n [g_j p_j(z) + \bar{g}_j \overline{p_j(z)}], \qquad g_j = a_j + ib_j, \tag{2.2}$$

(i is the imaginary unit) is also a real linear combination of the "circular harmonics" (2.1), and so is a harmonic polynomial. In fact, since z^j can be expressed (uniquely) as a complex linear combination of $p_0(z) \equiv 1$, $p_1(z),..., p_j(z)$, (j an integer, $0 \leqslant j \leqslant n$), any given harmonic polynomial of degree n has a unique representation of the type (2.2). The functions (2.1), restricted to Z_N, are linearly independent if and only if the functions

$$1, p_1(z) + \overline{p_1(z)},..., p_n(z)_n + \overline{p_n(z)}, i(p_1(z) - \overline{p_1(z)},..., i(p_n(z) - \overline{p_n(z)},$$

restricted to Z_N, are linearly independent, and a necessary and sufficient condition for this linear independence is that the functions

$$1, p_1(z),..., p_n(z), \overline{p_1(z)},..., \overline{p_n(z)},$$

restricted to Z_N, be linearly independent, using complex scalars.

Now suppose that the harmonics (2.1), restricted to Z_N, are indeed linearly independent. Let $u \in \mathscr{L}_C$ be given. Then according to the first paragraph in this section, there exists a unique real linear combination of the functions (2.1), say $H^{(0)}(u; z)$, which minimizes $\| H - u \|$ among all linear combinations of the functions (2.1). Also there exists a unique linear combination of the type (2.2) of the functions $1, p_1, ..., p_n, \bar{p}_1, ..., \bar{p}_n$, say $H^{(1)}(u; z)$, which minimizes $\| H - u \|$ among all such linear combinations. But $H^{(1)}(u; z)$ is, in fact, a real linear combination of the functions (2.1), so by the uniqueness it must be merely a rearrangement of $H^{(0)}(u; z)$, and $H^{(0)}(u; z) \equiv H^{(1)}(u; z)$ for all z.

In [3] it is shown that for each N, $N = 1, 2, ...$, there exists at least one set of points Z_N on the curve C such that the matrix M_N whose $(k + 1)$st row is $1, z_k, ..., z_k^N, \bar{z}_k, ..., \bar{z}_k^N$, $k = 0, 1, ..., 2N$, is nonsingular. Thus, its column vectors are linearly independent, and hence, if $n \leqslant N$, the functions $1, z, ..., z^n, \bar{z}, ..., \bar{z}^n$, restricted to Z_N, are, too, linearly independent. Thus, the assumption prerequisite to the existence of a unique harmonic polynomial of degree $n \leqslant N$ of best approximation in the above norm $\| \cdot \|$ is not vacuous, at least for some Z_N. Henceforth (as in the Introduction) a harmonic polynomial of degree at most n of best approximation to u in the norm $\| \cdot \|$ will be denoted by $H_n(Z_N; u; z)$.

Now let Z_N be such that the harmonics (2.1), restricted to Z_N, are linearly independent, so that $H_n(Z_N; u; z)$ exists and is unique. In terms of linear approximation in the real normed linear space $\mathscr{L}_C$, if H_n is based, as in (2.2), on the polynomials $1, p_1, ..., p_n$, we should regard H_n as the best linear approximation to u on Z_N with the elements $1, p_1 + \bar{p}_1, p_2 + \bar{p}_2, ..., p_n + \bar{p}_n$, $i(p_1 - \bar{p}_1), ..., i(p_n - \bar{p}_n)$. The normal equations [7, p. 176] for determining the coefficients g_0, a_j, b_j in (2.2) are

$$\sum_{k=0}^{2N} [H_n(Z_N; u; z_k) - u(z_k)] = 0,$$

$$\sum_{k=0}^{2N} [p_j(z_k) + \overline{p_j(z_k)}][H_n(Z_N; u; z_k) - u(z_k)] = 0,$$

$$\sum_{k=0}^{2N} i[p_j(z_k) - \overline{p_j(z_k)}][H_n(Z_N; u; z_k) - u(z_k)] = 0,$$

$$j = 1, 2, ..., n.$$

It is convenient, for theoretical purposes, to use an equivalent complex form for these equations, which is obtained by using the third member of (2.2) and by obvious elementary transformations:

$$\sum_{k=0}^{2N}\left[g_0+\sum_{j=1}^{n}g_jp_j(z_k)+\sum_{j=1}^{n}\bar{g}_j\overline{p_j(z_k)}\right]=\sum_{k=0}^{2N}u(z_k),$$

$$\sum_{k=0}^{2N}\overline{p_J(z_k)}\left[g_0+\sum_{j=1}^{n}g_jp_j(z_k)+\sum_{j=1}^{n}\bar{g}_j\overline{p_j(z_k)}\right]=\sum_{k=0}^{2N}\overline{p_J(z_k)}\,u(z_k), \tag{2.3}$$

$$J=1,2,\ldots,n;$$

$$\sum_{k=0}^{2N}p_J(z_k)\left[g_0+\sum_{j=1}^{n}g_jp_j(z_k)+\sum_{j=1}^{n}\bar{g}_j\overline{p_jz_k}\right]=\sum_{k=0}^{2N}p_J(z_k)\,u(z_k).$$

$$J=1,2,\ldots,n.$$

Let P_{Nn} be the $(2N+1)\times(2n+1)$ matrix

$$\begin{bmatrix}1, p_1(z_k),\ldots, p_n(z_k), \overline{p_1(z_k)},\ldots, \overline{p_n(z_k)}\\ k=0,1,\ldots,2N\end{bmatrix}$$

Let $\mathbf{g}_n=(g_0, g_1,\ldots, g_n, \bar{g}_1,\ldots, \bar{g}_n)$ and $\mathbf{u}_N=(u(z_0), u(z_1),\ldots, u(z_{2N}))$. The normal Eqs. (2.3) in matrix form become

$$P_{Nn}^*P_{Nn}\mathbf{g}_n=P_{Nn}^*\mathbf{u}_N, \tag{2.4}$$

where the asterisk means conjugate transpose. The matrix $P_{Nn}^*P_{Nn}$ is related by elementary transformations to the Gram matrix [7, p. 177] of the elements $1, p_1+\bar{p}_1,\ldots, i(p_1-\bar{p}_1),\ldots$, and so must be nonsingular under the assumption that these harmonics are linearly independent on Z_N. Conversely, if the matrix $P_{Nn}^*P_{Nn}$ is nonsingular, then it is easy to see by elementary linear algebra that the columns of P_{Nn} are linearly independent, and $H_n(Z_N; u; z)$ exists uniquely, with coefficients g_n given by (2.4).

3. Convergence Theory: Formulation in Terms of Faber Polynomials

Given a double sequence of complex numbers $\langle a_{N,n}, n<N\rangle_{N,n=1}^{\infty}$, we shall say that $\lim_{n\to\infty} a_{N,n}=a$, uniformly in $N>n$, if, given any $\epsilon>0$, there exists an $n_\epsilon\geq 1$ such that $|a_{N,n}-a|<\epsilon$ for all $n\geq n_\epsilon$ and all $N>n$. If $a_{N,n}=a_{N,n}(z)$ is a value of a function with domain $\mathscr{B}$, the statement $\lim_{n\to\infty} a_{N,n}(z)=a(z)$, uniformly in $N>n$ and almost uniformly on $\mathscr{B}$ means that given any $\epsilon>0$ and any compact subset $\mathscr{B}_0$ of $\mathscr{B}$, there exists an $n_\epsilon>0$ (which may depend on $\mathscr{B}_0$) such that $|a_{N,n}(z)-a(z)|<\epsilon$ for all $n\geq n_\epsilon$, all $N>n$, and all z on $\mathscr{B}_0$. Our basic convergence theorem is

THEOREM 3.1. *Let C be an analytic simple closed curve, and let $u(z)$ be continuous on C. There exist sequences $S = \langle Z_N \rangle_1^\infty = \langle \{z_{N0}, z_{N1}, \ldots, z_{N,2N}\} \rangle_1^\infty$ of subsets of C such that:*

(1) *For some positive integer n_S there is a uniquely determined least-squares polynomial $H(Z_N; u; z)$ with coefficients given by* (2.4), *for each pair N, n with $N > n \geqslant n_S$.*

(2) $\mathrm{Lim}_{n\to\infty} \int_C | H_n(Z_N; u; z) - u(z)|^2 \, | dz | = 0$, *uniformly in $N > n$.*

(3) $\mathrm{Lim}_{n\to\infty} H_n(Z_N; u; z) = U(z)$, *uniformly in $N > n$ and almost uniformly on* Int C, *the interior region of C, where $U(z)$ is the solution of the Dirichlet problem for u and* Int C.

The statement (1) is equivalent to saying that for $N > n \geqslant n_S$, the matrix $P_{Nn}^* P_{Nn}$ is nonsingular.

To characterize successful sequences S we shall follow a tradition in complex polynomial interpolation theory in which the problem of proper spacing on C is referred, by conformal mapping, to the unit circle. (See [4, 5, 8].) Given any simple closed curve C, there exists a function

$$z = \phi(w) = d\left(w + d_0 + \frac{d_1}{w} + \frac{d_2}{w^2} + \cdots\right), \qquad d > 0, \qquad (3.1)$$

which is univalent and analytic for $| w | > 1$, and which maps $\{w: | w | > 1\}$ conformally onto Ext C (the exterior region of C). The number d is the transfinite diameter (capacity) of C. The function ϕ has a continuous extension onto $\Gamma = \{w: | w | = 1\}$ which gives a topological mapping of Γ onto C. Given any set $\{z_{N0}, z_{N1}, \ldots, z_{N,2N}\}$ of $2N + 1$ (distinct) points on C, we propose to characterize its distribution on C by that of the image points $w_{Nk} = \phi^{-1}(z_{Nk})$, $k = 0, 1, \ldots, 2N$ on the unit circle in the w-plane.

The discussion in Section 2 shows that in the presence of uniqueness of the least-squares solution, it makes no difference how we choose the polynomials p_j in the representation (2.2) insofar as the values of $H(Z_N; u; z)$ for a given Z_N and u are concerned, provided that the polynomials p_j are of respective degrees exactly j. For computational purposes, one might well wish to choose $p_j(z) = z^j$. But since the convergence program here involves referral to the unit circle via the mapping (3.1), it is natural to select a set of base polynomials p_j which assume a reasonably simple form as functions of w after this transformation. Such a property is possessed by the Faber polynomials associated with C. (See [6], where an extensive bibliography is given). We develop briefly those formal properties of these polynomials which are essential for present purposes.

Let C be an arbitrary simple closed curve. For z exterior to a sufficiently large circle, the inverse transformation of (3.1) has a Laurent expansion

$$\phi^{-1}(z) = \frac{z}{d} + c_0 + \sum_{k=1}^{\infty} c_k z^{-k}.$$

The jth Faber polynomial $p_j(z)$, $j = 1, 2, \ldots$, belonging to C (or to ϕ) is the principal part at infinity of the Laurent expansion of $[\phi^{-1}(z)]^j$. Clearly the coefficient of z^j in p_j is $d^{-j} \neq 0$. These polynomials can be calculated from the recursion formulas

$$p_1(z) = (z/d) - d_0\,,$$

$$p_{j+1}(z) = p_1(z)\,p_j(z) - d_1 p_{j-1} - d_2 p_{j-2}(z) - \cdots - d_{j-1} p_1(z) - (j+1)\,d_j\,, \qquad j = 1, 2, \ldots\,.$$

(In using this, it is to be understood that $p_0 = p_{-1} = p_{-2} = \cdots = 0$.) It is easy to show by the calculus of residues [6] that after the transformation $z = \phi(w)$, p_j assumes the form

$$p_j(\phi(w)) = w^j + \sum_{k=1}^{\infty} \alpha_{jk} w^{-k}, \qquad j = 1, 2, \ldots, \; |w| > 1. \tag{3.2}$$

Each of the series here converges almost everywhere on $|w| = 1$ in the Lebesgue sense, and if C is rectifiable (as in Theorem 3.1) the series converges absolutely for $|w| = 1$. (See [6, Theorems 4.3 and 4.4]). The numbers α_{jk} are called the Faber coefficients for C (or ϕ).

Now let $Z_N = \{Z_{N0}, \ldots, z_{N,2N}\}$ be any set of $2N + 1$ (distinct) points on an *arbitrary* simple closed curve C, and let $W_N = \{w_{N0}, w_{N1}, \ldots, w_{N,2N}\}$ be the image set on $|w| = 1$ under the transformation ϕ^{-1}, determined by (3.1).

After the substitution $z = \phi(w)$, the matrix P_{Nn} of Section 2 becomes

$$P_{Nn} = \begin{bmatrix} 1,\, w_{Nk} + \sum_{h=1}^{\infty} \alpha_{1h} \bar{w}_{Nk}^h, \ldots, w_{Nk}^n + \sum_{h=1}^{\infty} \alpha_{nh} \bar{w}_{Nk}^h,\, \bar{w}_{Nk} + \sum_{h=1}^{\infty} \alpha_{1h} w_{Nk}^h, \ldots, \\ \bar{w}_{Nk}^n + \sum_{h=1}^{\infty} \alpha_{nh} w_{Nk}^h \\ k = 0, 1, 2, \ldots, 2N \end{bmatrix}.$$

Let matrices A_n, U_{Nn}, and E_{Nn} be defined as follows:

$$A_n = \left[\begin{array}{ccccc|cccc} & & & & & 0 & 0 & \cdots & 0 \\ & & I_{n+1} & & & \bar{\alpha}_{11} & \bar{\alpha}_{21} & \cdots & \bar{\alpha}_{n1} \\ & & & & & \vdots & \vdots & & \vdots \\ & & & & & \bar{\alpha}_{n1} & \bar{\alpha}_{2n} & \cdots & \bar{\alpha}_{nn} \\ \hline 0 & \alpha_{11} & \alpha_{21} & \cdots & \alpha_{n1} & & & & \\ 0 & \alpha_{12} & \alpha_{22} & \cdots & \alpha_{n2} & & & I_n & \\ \vdots & \vdots & \vdots & & \vdots & & & & \\ 0 & \alpha_{1n} & \alpha_{2n} & \cdots & \alpha_{nn} & & & & \end{array}\right],$$

where I_M is the identity matrix of order M;

$$U_{Nn} = \begin{bmatrix} 1, w_{Nk}, w_{Nk}^2, \ldots, w_{Nk}^n, \bar{w}_{Nk}, \ldots, \bar{w}_{Nk}^n \\ k = 0, 1, 2, \ldots, 2N \end{bmatrix},$$

$$E_{Nn} = \begin{bmatrix} 0, \sum_{h=n+1}^{\infty} \alpha_{1h}\bar{w}_{Nk}^h, \ldots, \sum_{h=n+1}^{\infty} \alpha_{nh}\bar{w}_{Nk}^h, \sum_{h=n+1}^{\infty} \bar{\alpha}_{nh} w_{Nk}^h, \ldots, \sum_{h=n+1}^{\infty} \bar{\alpha}_{nh} w_{Nk}^h \\ k = 0, 1, 2, \ldots, 2N. \end{bmatrix}.$$

Then $P_{Nn} = U_{Nn}A_n + E_{Nn}$, and the normal equations (2.4) become

$$(A_n{}^*U_{Nn}^* + E_{Nn}^*)(U_{Nn}A_n + E_{Nn})\,\mathbf{g} = A_n{}^*U_{Nn}^*\mathbf{u}_N + E_{Nn}^*\mathbf{u}_N. \qquad (3.3)$$

We now introduce a vector whose components are Fourier–Lagrange coefficients for $u[\phi(w)]$ relative to the set $W_N = \{w_{Nk}, h = 0,\ldots, 2N\}$. (The terminology is that of [21, Chap. 10].) For any integer h, let $l_{Nh} = (2N+1)^{-1}\sum_{k=0}^{2N} \bar{w}_{Nk}^h u[\phi(w_{Nk})]$, and let $l_n^{(N)} = (l_{N0}, l_{N1}, \ldots, l_{Nn}, \bar{l}_{N1}, \bar{l}_{N2}, \ldots, \bar{l}_{Nn})$. Then $U_{Nn}^*\mathbf{u}_N = (2N+1)\, l_n^{(N)}$, and (3.3) becomes

$$(A_n{}^*U_{Nn}^* + E_{Nn}^*)(U_{Nn}A_n + E_{Nn})\,\mathbf{g}_n = (2N+1)\,A_n{}^*l_n^{(N)} + E_{Nn}^*\mathbf{u}_N. \qquad (3.4)$$

4. Convergence Theory: Restrictions on the Spacing of the Nodes

We shall make frequent use of the row norm of a matrix as a uniform measure of the magnitude of its elements. Given an arbitrary $m \times n$ complex matrix $B = [b_{ij}]$, the row norm is $\rho(B) = \max_i \sum_{j=1}^n |b_{ij}|$. If B_1 and B_2 are two $m \times n$ matrices, then $\rho(B_1 + B_2) \leqslant \rho(B_1) + \rho(B_2)$. If B_1 is an $m \times p$ matrix and B_2 is a $p \times n$ matrix, then $\rho(B_1B_2) \leqslant \rho(B_1)\,\rho(B_2)$. Vectors will be regarded as single-column matrices, so that these norm inequalities will be available when sums of vectors and products of matrices with vectors are considered. Of course, the row norm of a vector in this treatment is simply the maximum of the absolute values of its components.

For the time being, we shall still suppose that C is an arbitrary simple closed curve. As was implied in Section 3, we shall impose spacing conditions on the sequence $\langle Z_N\rangle_{N=1}^{\infty} = \langle\{Z_{N0},..., Z_{N,2N}\}\rangle_{N=1}^{\infty}$ of sets Z_N of distinct points on C by placing requirements on the sequence of sets W_N of image points, where $W_N = \{w_{Nk} = \phi^{-1}(z_{Nk}): k = 0,..., 2N\}$, with ϕ defined in (3.1). The matrix U_{Nn} introduced in Section 3, whose $(k+1)$th row is $(1, w_{Nk},..., w_{Nk}^n, \bar{w}_{Nk},..., \bar{w}_{Nk}^n)$, $k = 0,..., 2N$, will play a key role.

A preliminary remark is in order concerning $U_{Nn}^*U_{Nn}$. By writing $\bar{w}_{Nk} = w_{Nk}^{-1}$, the determinant of U_{NN} can be identified as a Vandermonde determinant multiplied by a nonzero complex number (the explicit formula is given in [21, Vol. 2, p. 1]), and so, if the points w_{Nk}, $k = 0,..., 2N$ are distinct, then U_{NN} is nonsingular. Thus, its columns are linearly independent, and so the columns of U_{Nn}, $n < N$, are also linearly independent. It follows from elementary linear algebra that $U_{Nn}^*U_{Nn}$ is *always* nonsingular, for any choice of Z_N in which the points are distinct.

We now identify four distribution hypotheses:

(I) There exists an absolute constant μ such that $\rho[(U_{Nn}^*U_{Nn})^{-1}] \leqslant (2N+1)^{-1}\mu$ for all n, N, $N > n \geqslant 1$.

(II) $\lim_{n\to\infty} n^{1/2}\rho[(2N+1)(U_{Nn}^*U_{Nn})^{-1} - I_{2n+1}] = 0$, uniformly in $N > n$.

(III) The sequence of sets $\langle\{w_{N0}, w_{N1},..., w_{N,2N}\}\rangle_{N=1}^{\infty}$ is strongly equidistributed on the unit circle, in a sense to be defined below.

(IV) $\lim_{N\to\infty} \rho[(2N+1)^{-1} U_{NN}U_{NN}^* - I_{2N+1}] = 0$.

To define the equidistribution property in (III), let $w_{Nk} = \exp(i\theta_{Nk})$, $0 \leqslant \theta_{Nk} < 2\pi$; let $\nu_N(\theta)$ be the number of points w_{Nk} in the set $\{w_{N0}, w_{n1},..., w_{Nk}\}$ with arguments θ_{Nk} not exceeding θ. The condition is that $\lim_{N\to\infty} \nu_N(\theta)/N = \theta/2\pi$ for each θ, $0 \leqslant \theta < 2\pi$. *It is known that a necessary and sufficient condition for the sequence of sets to be equidistributed in this sense is that for every real-valued function* $f(\theta)$, *piecewise continuous on* $[0, 2\pi]$, *we have* $\lim_{N\to\infty}(2N+1)^{-1}\sum_{k=0}^{2N} f(\theta_{Nk}) = (2\pi)^{-1}\int_0^{2\pi} f(\theta)\, d\theta$ (See [19, pp. 164–165]).

It is easy to see that (II) implies (I), but the relationships between the other conditions are not so clear. However, let $w_{Nk} = \exp[2\pi ik(2N+1)^{-1}]$, $k = 0, 1,..., 2N$, so that the w_{Nk}'s are the $(2N+1)$th roots of unity. We have:

$$w_{Nk}^j\bar{w}_{Nk}^j = 1, \qquad \text{for all integers } j \text{ and } k;$$

$$\sum_{k=0}^{2N} w_{Nk}^j = \sum_{k=0}^{2N} \bar{w}_{Nk}^j = \begin{cases} 0 & \text{if } j \not\equiv 0 \pmod{2N+1}, \\ 2N+1 & \text{if } j \equiv 0 \pmod{2N+1}. \end{cases} \tag{4.1}$$

Then

$$U_{Nn}^* U_{Nn} = (2N+1)\, I_{2n+1}, \qquad (U_{Nn}^* U_{Nn})^{-1} = (2N+1)^{-1} I_{2n+1},$$
$$U_{NN} U_{NN}^* = (2N+1)\, I_{2N+1},$$

and (II) and (IV) are satisfied trivially. Also, by the criterion in italics above, the sequence of sets of roots of unity satisfies (III). What the conditions (I)–(IV), are saying, then, is that the distribution of $\{w_{Nk} : k = 0,..., 2N\}$, for each N, is not too far from an equally spaced distribution.

It might be of interest to look more closely into the geometric meaning of conditions (II) and (IV).

5. Convergence Theory: Row Norm Inequalities

We return to the normal equations in the form (3.4) for determining $H_n(Z_N\,; u; z)$. The problem of the unisolvability and convergence of the solution vector of the normal equations remains open when C is not an analytic curve, even when $W_N = \phi^{-1}(Z_N)$ consists of the $(2N+1)$th roots of unity for each N. The main problem is that of the nonsingularity of the matrix of A_n of Section 3, which has been established only when C is analytic.

If C is an analytic simple closed curve, then there exists a number r, $0 < r < 1$, such that ϕ can be continued across $|w| = 1$ so as to become univalent and analytic for $|w| \geqslant r$. The Laurent series (2.3) for the Faber polynomials then converge absolutely for $|w| \geqslant r$. The number r (which is not necessarily minimal) plays a key role in the ensuing analysis. In the first place, we have the following estimates, due to Grunsky [9] and Pommerenke [13]:

$$|\alpha_{jh}| \leqslant \left(\frac{j}{h}\right)^{1/2} r^{j+h}, \qquad j, h = 1, 2,... . \tag{5.1}$$

(Actually the more naive estimate $\alpha_{jh} = O(r_0^{j+h})$, $r < r_0 < 1$, derivable from the Cauchy coefficient inequalities, is all that will be needed explicitly in the present paper.) In the second place, and this is more important, the matrix A_n of Section 3 is nonsingular for each n. This was proved by the author in [5], with the help of a sharper form of the Grunsky–Pommerenke inequalities [6, 9, 10]. The basic result [5, Theorem 2.1] is

Theorem 5.1. *Let* α_{jh}, j, $h = 1, 2,...$, *be the Faber coefficients for the function* $z = \phi(w)$, *analytic and univalent for* $|w| \geqslant r$, $0 < r < 1$. *Let* $a_0, a_1,...,$ *and* $b_1, b_2,...$ *be any two given infinite sequences of complex numbers. Let* $\mathbf{a}_n$ *be the vector* $(a_0, a_1,..., a_n,..., b_n)$ *and let* $\mathbf{x}_n$ *be the vector*

$(a_0, x_{n1}, \ldots, x_{nn}, y_{n1}, \ldots, y_{nn})$. *The equation* $A_n \mathbf{x}_n = \mathbf{a}_n$ *has a unique solution* $\mathbf{x}_n$ *which satisfies the following inequalities, in which the right members are independent of* n:

$$
\begin{aligned}
| x_{nk} | &\leqslant \frac{M_a r}{(1 - r^2) k^{1/2}} + | a_k |, \qquad k = 1, 2, \ldots, n, \\
| y_{nk} | &\leqslant \frac{M_a r}{(1 - r^2) k^{1/2}} + | b_k |, \qquad k = 1, 2, \ldots, n,
\end{aligned}
\tag{5.2}
$$

where $M_a^2 = \max(\sum_{k=1}^{\infty} kr^{2k} | a_k |^2, \sum_{k=1}^{\infty} kr^{2k} | b_k |^2)$.

Henceforth in this paper, it will always be assumed that C is an analytic curve and that r, $0 < r < 1$, is such that ϕ is univalent and analytic for $| w | \geqslant r$.

We shall need estimates for the row norms of various matrices and vectors associated with (3.4). In most cases, the proofs are obvious consequences of (5.1) and (5.2), and will be omitted. The numbers a_r, b_r, α_r, etc., which appear in the following estimates are positive constants which depend on r, but not on n, N, nor on C, otherwise than through r. Typically they become infinite as $r \to 1$. The notation at the right, in parentheses, gives the inequality or equation used to derive the estimate.

(5.3) $$\rho(A_n) \leqslant a_r, \qquad (5.1)$$

(5.4) $$\rho(A_n^{-1}) \leqslant b_r, \qquad (5.2)$$

(5.5) $$\rho[(A_n{}^*)^{-1}] \leqslant c_r n. \qquad (5.2)$$

(This is of course the column norm of A_n^{-1}, and (5.4) and (5.5) are both proved by using unit vectors a_n in (5.2)).

There are no restrictions on W_N in the following estimates:

(5.6) $$\rho(U_{Nn}) \leqslant 2n + 1 \qquad (| w_{Nk} | = 1),$$

(5.7) $$\rho(U_{Nn}^*) \leqslant 2N + 1 \qquad (| w_{Nk} | = 1),$$

(5.8) $$\rho(E_{Nn}) \leqslant d_r r^n, \qquad (5.1)$$

(5.9) $$\rho(E_{Nn}^*) \leqslant (2N + 1)\, d_r' r^n. \qquad (5.1)$$

An unpleasant looking $(2n + 1) \times (2n + 1)$ matrix

$$\mathcal{A}_n = -U_{Nn}^* E_{Nn} - (A_n{}^*)^{-1} E_{Nn}^* (U_{Nn} A_n + E_{Nn})$$

will enter the picture, and we estimate its row norm by the inequalities in the first paragraph of Section 4.

In terms of the above constants,

$$\rho(\mathfrak{A}_n) \leqslant (2N+1)\, d_r r^n + c_r n(2N+1)\, d_r' r^n((2n+1)\, a + dr^n),$$

which we condense to

$$\rho(\mathfrak{A}_n) \leqslant (2N+1)\, \alpha_r n^2 r^n. \tag{5.10}$$

Finally, if $u(z)$ is continuous on C, we let $M = \max |u(z)|$, z on C. From the definition of $\mathbf{l}_n^{(N)}$, we have

$$\rho(\mathbf{l}_n^{(N)}) \leqslant M, \qquad n = 1, 2, \dots, N, \quad N = 1, 2, 3, \dots . \tag{5.11}$$

6. Convergence Theory: Proof of Theorem 3.1

Proof. We assume that C is an analytic simple closed curve, the mapping ϕ of (3.1) is analytic and univalent for $|w| \geqslant r$, $r < 1$, $u(z)$ is continuous on C, and, for any set Z_N on C, $H_n(Z_N; u; z)$ (if it exists) is written in the form (2.2) in which p_j is the jth Faber polynomial belonging to ϕ. To emphasize the dependence of the coefficient vector on N, as well as on n, we use the notation $\mathbf{g}_n = \mathbf{g}_n^{(N)} = (g_{N0}, g_{N1}, \dots, g_{Nn}, \bar{g}_{N1}, \dots, \bar{g}_{Nn})$.

We premultiply the normal equations (2.4) by $(A_n{}^* U_{Nn}^* U_{Nn})^{-1} = (U_{Nn}^* U_{Nn})^{-1} (A_n{}^*)^{-1}$, and obtain, after some rearrangement,

$$\begin{aligned} A_n \mathbf{g}_n^{(N)} &= (2N+1)(U_{Nn}^* U_{Nn})^{-1} \mathbf{l}_n^{(N)} + (U_{Nn}^* U_{Nn})^{-1} (A_n{}^*)^{-1} E_{Nn}^* \mathbf{u}_N \\ &\quad + (U_{Nn}^* U_{Nn})^{-1} \mathfrak{A}_n \mathbf{g}_n^{(N)}, \end{aligned} \tag{6.1}$$

where $\mathfrak{A}_n$ is the matrix introduced in Section 5. (The inverses displayed here exist for all $n \leqslant N$, $N = 1, 2, \dots$, by Theorem 5.1 and the remark concerning $U_{Nn}^* U_{Nn}$ in Section 4.) Another form of the equation, obtained from (6.1) by premultiplication by A_n^{-1}, is

$$\begin{aligned} (I_{2n+1} - A_n^{-1}(U_{Nn}^* U_{Nn})^{-1} \mathfrak{A}_n)\, \mathbf{g}_n^{(N)} &= A_n^{-1}(U_{Nn}^* U_{Nn})^{-1}[(2N+1)\, \mathbf{l}_n^{(N)} \\ &\quad + (A_n{}^*)^{-1} E_{Nn}^* \mathbf{u}_N] \end{aligned} \tag{6.2}$$

Lemma 6.1. *Let the sequence* $\langle W_N \rangle$ *satisfy condition* (I) *of Section* 4. *Then*

(a) *there exists an* $n_0 > 0$ *such that for all* $N > n \geqslant n_0$, *the normal equations are unisolvable, and*

(b) *there exists a constant* g_r *depending only on the function* $u(z)$ *and on* r, *but not on* N *and* n, *such that* $\rho(\mathbf{g}_n^{(N)}) \leqslant g_r$ *for all* n, N, $N > n \geqslant n_0$.

To prove (a), we use (6.2). From (5.4), condition (I), and (5.10), we have

$$\begin{aligned}\rho_n &= \rho[A_n^{-1}(U_{Nn}^*U_{Nn})^{-1}\mathcal{C}\!\ell_n] \leqslant \rho(A_n^{-1})\,\rho[(U_{Nn}^*U_{Nn})^{-1}]\,\rho(\mathcal{C}\!\ell_n)\\ &\leqslant b_r(2N+1)^{-1}\,\mu(2N+1)\,\alpha_r n^2 r^n = b_r\mu\alpha_r n^2 r^n < \tfrac{1}{2} \qquad \text{for all} \quad n \geqslant n_0\,,\end{aligned}$$

with n_0 suitably chosen. The sum of the absolute values of the off-diagonal terms in any given row of $K_n = I_{2n+1} - A_n^{-1}(U_{Nn}^*U_{Nn})^{-1}\,\mathcal{C}\!\ell_n$ is less than ρ_n, and so is less than 1/2 for $n \geqslant n_0$. However, the term in this row of K_n on the diagonal is greater than or equal to $1 - \rho[A_n^{-1}(U_{Nn}^*U_{Nn})^{-1}\,\mathcal{C}\!\ell_n]$, and so is greater than 1/2 for $n \geqslant n_0$. Thus, for $n \geqslant n_0$, the matrix K_n has a dominant diagonal. Then a well-known theorem [15] assures that K_n is nonsingular for all $n \geqslant n_0$, and this proves part (a) of the Lemma.

For (b), we transpose the term $A_n^{-1}(U_{Nn}^*U_{Nn})^{-1}\,\mathcal{C}\!\ell_n\mathbf{g}_n^{(N)}$ in (6.2) to the right side. We then estimate the row norm of $I_{2n+1}\mathbf{g}_n^{(N)}$ by the general row norm inequalities (Section 4, first paragraph), with the aid of the library of row norm equalities in Section 5 [including (5.11)], and the estimate in the preceding paragraph. With the estimate for the transposed term appearing first, for $n \geqslant n_0$ we have

$$\rho(\mathbf{g}_n^{(N)}) \leqslant \tfrac{1}{2}\rho(\mathbf{g}_n^{(N)}) + b_r(2N+1)^{-1}\mu[(2N+1)\,M + c_r n(2N+1)\,d_r' r^n M],$$

and (b) follows from this inequality after subtracting $\rho(g_n^{(N)})/2$ from both sides.

It will be noted that Lemma (6.1) (a) proves part (1) of Theorem 3.1 provided that the sequence of point sets S mentioned in the theorem satisfies condition (I). The integer n_S in the theorem is the n_0 of Lemma 6.1 (a), and henceforth will be so identified.

We now establish part (2) of Theorem 3.1 by proving

THEOREM 6.1. *If the sequence* $\langle W_N\rangle$ *satisfies conditions* (II)–(IV), *then*

$$\lim_{n\to\infty}\int_0^{2\pi} |\, H_n[Z_N\,;u;\,\phi(w)] - u[\phi(w)]\,|^2\,d\theta = 0$$

uniformly in $N > n \geqslant n_S$, *where* $w = \exp(i\theta)$.

Let the formal trigonometric Fourier series for $H_n[Z_N\,;u;\,\phi(w)] - u(\phi(w)]$ be denoted by $\sum_{-\infty}^{\infty} F_h w^h$, where $w = \exp(i\theta)$. By substituting (3.2) into (2.2), we obtain the following representation of H_n:

$$\begin{aligned}H_n(Z_N\,;u,\,\phi(w)) = g_{N0} &+ \sum_{j=1}^{n} g_{Nj}\left(w^j + \sum_{h=1}^{\infty}\alpha_{jh}w^{-h}\right)\\ &+ \sum_{j=1}^{n}\overline{g_{Nj}}\left(\bar{w}^j + \sum_{h=1}^{\infty}\bar{\alpha}_{jh}w^{-h}\right),\end{aligned}$$

valid for $|w| \geqslant r$; for as stated in Section 3, each of the infinite series displayed here converges for $|w| \geqslant r$. With $w = \exp(i\theta)$, rearrangement to form the Fourier series for H_n is permissible. We introduce the Fourier series for $u[\phi(w)]$ in complex form:

$$u[\phi(w)] \sim \sum_{h=-\infty}^{\infty} f_h w^h, \qquad w = \exp(i\theta),$$

$$f_h = \frac{1}{2\pi} \int_0^{2\pi} \bar{\xi}^h u[\phi(\xi)]\, dt, \qquad \xi = \exp(it).$$

By picking out coefficients of like powers of w in $H_n - u$, with $|w| = 1$ (so that $\bar{w} = w^{-1}$), we find that $F_0 = g_{N0} - f_0$, $F_h = g_{Nh} - f_h + \sum_{j=1}^n \bar{\alpha}_{jh}\bar{g}_{Nj}$, $h = 1, 2,..., n$, $F_h = -f_h + \sum_{j=1}^n \bar{\alpha}_{jh}\bar{g}_{Nj}$, $h = n+1, n+2,...$. Also $F_{-h} = \bar{F}_h$, $h = 1, 2,...$. Parseval's formula [21, Vol. 1, p. 37] for $H_n - u$ takes the form

$$\frac{1}{2\pi} \int_0^{2\pi} | H_n[Z_n ; u; \phi(\exp(i\theta))] - u[\phi(\exp(i\theta))] |^2 \, d\theta$$

$$= \sum_{-\infty}^{\infty} | F_h |^2 = \delta_{Nn} + \epsilon_{Nn}, \tag{6.3}$$

where

$$\delta_{Nn} = | g_{N0} - f_0 |^2 + 2 \sum_{h=1}^{n} \left| g_{Nh} - f_h + \sum_{j=1}^{n} \bar{\alpha}_{jh}\bar{g}_{Nj} \right|^2,$$

$$\epsilon_{Nn} = 2 \sum_{h=n+1}^{\infty} \left| -f_h + \sum_{j=1}^{n} \bar{\alpha}_{jh}\bar{g}_{Nj} \right|^2.$$

We shall prove Theorem 6.1 by showing that $\lim_{n\to\infty} \epsilon_{Nn} = \lim_{n\to\infty} \delta_{Nn} = 0$, uniformly in $N > n \geqslant n_S$.

LEMMA 6.2. *Let the sequence $\langle W_N \rangle$ satisfy condition* (II) *of Section* 4. *Then*

(a) $\lim_{N\to\infty} | g_{N,0} - l_{N,0} | = 0$;

(b) $\lim_{n\to\infty} \sum_{h=1}^n | g_{Nh} + \sum_{j=1}^n \bar{\alpha}_{jh}\bar{g}_{Nj} - l_{Nh} |^2 = 0$, *uniformly in* $N > n \geqslant n_S$.

For the proof, we observe that $g_{Nh} + \sum_{j=1}^n \bar{\alpha}_{jh}\bar{g}_{Nj} - l_{Nh}$ is the $(h+1)$nth component of the vector $A_n \mathbf{g}_n^{(N)} - \boldsymbol{l}_n^{(N)}$, $h = 1,..., n$, and $g_{N,0} - l_{N,0}$ is the first component. We have

$$| g_{N0} - l_{N0} |^2 + \sum_{h=1}^{n} \left| g_{Nk} + \sum_{j=1}^{n} \bar{\alpha}_{jh}\bar{g}_{Nj} - l_{Nk} \right|^2$$

$$\leqslant (n+1)[\rho(A_n \mathbf{g}_n^{(N)} - \boldsymbol{l}_n^{(N)})]^2 = [\rho(\sqrt{(n+1)}^{1/2}(A_n \mathbf{g}_n^{(N)} - \boldsymbol{l}_n^{(N)}))]^2,$$

and we shall show that the last member tends to zero, uniformly in $N > n$. Using (6.1), we have

$$
\begin{aligned}
&(n+1)^{1/2}(A_n \mathbf{g}_n^{(N)} - \boldsymbol{l}_n^{(N)}) \\
&\quad = (1+n^{-1})^{1/2} n^{1/2} [(2N+1)(U_{Nn}^* U_{Nn})^{-1} - I_{2n+1}]\, \boldsymbol{l}_n^{(N)} \\
&\qquad + (n+1)^{1/2} [(U_{Nn}^* U_{Nn})^{-1} (A_n{}^*)^{-1} E_{Nn}^* \mathbf{u}_N + (U_{Nn}^* U_{Nn})^{-1} \mathcal{C}_n \mathbf{g}_n^{(N)}].
\end{aligned}
$$

It was noted in Section 4, in the discussion of conditions (I–IV), that (II) implies (I); and we here assume that the constant μ in (I) is taken to accord with this implication. Using the library of row norm inequalities in Section 5, we quickly arrive at the estimate

$$
\begin{aligned}
&\rho[(n+1)^{1/2}(A_n \mathbf{g}_n^{(N)} - \boldsymbol{l}_n^{(N)})] \\
&\quad \leqslant (1+n^{-1})^{1/2} n^{1/2} \rho[(2N+1)(U_{Nn}^* U_{Nn})^{-1} - I_{2n+1})] M \\
&\qquad + (n+1)^{1/2} [(2N+1)^{-1} \mu c_r n (2N+1)\, d_r' r^n \cdot M \\
&\qquad + (2N+1)^{-1} (2N+1)\, \alpha_r n^2 r^n g_r],
\end{aligned}
$$

in which g_r comes from Lemma 6.1 (b). The assumption of condition (II) takes care of the convergence to zero of the first term on the right, and the other term is obviously $O(n^{5/2} r^n)$. This concludes the proof of Lemma 6.2.

The next four lemmas are concerned with the Fourier–Lagrange coefficients of u, with particular reference to their relation with the Fourier coefficients of u.

LEMMA 6.3. *Let the sequence $\langle W_N \rangle$ satisfy condition* (III) *of Section* 4. *Then*

(a) *for each fixed h, $h = 0, 1, 2, \ldots$, $\lim_{N\to\infty} l_{Nh} = f_h$;*
(b) $\lim_{N\to\infty} (2N+1)^{-1} \sum_{k=0}^{2N} (u[\phi(w_{Nk})])^2 = (2\pi)^{-1} \int_0^{2\pi} (u[\phi(\exp(it))])^2\, dt.$

The Lemma follows at once from the fact that u, u^2, and $\bar{w}^h$ are all continuous functions, so that the necessary (and sufficient) condition for an equidistribution given in Section 4, in the discussion of (III), is available.

LEMMA 6.4. *Let $\langle W_N \rangle$ satisfy conditions* (III) *and* (IV) *of Section* 4. *Then*

$$
\lim_{N\to\infty} \sum_{h=-N}^{N} | l_{Nh} |^2 = \frac{1}{2\pi} \int_0^{2\pi} (u[\phi(\exp(it))])^2\, dt.
$$

(This result could be called Parseval's formula for the Fourier–Lagrange coefficients l_{Nh}.)

By definition, $l_N^{(N)} = (2N+1)^{-1} U_{NN}^* \mathbf{u}_N$.
We have

$$\begin{aligned}
&\Big| \sum_{k=-N}^{N} | l_{Nh} |^2 - \frac{1}{2N+1} \sum_{k=0}^{2N} (u[\phi(w_{Nh})])^2 \Big| \\
&= \Big| l_N^{(N)*} l_N^{(N)} - \frac{1}{2N+1} \mathbf{u}_N{}^* \mathbf{u}_N \Big| \\
&= \Big| \frac{1}{(2N+1)^2} \mathbf{u}_N{}^* U_{NN} U_{NN}^* \mathbf{u}_N - \frac{1}{(2N+1)} \mathbf{u}_N{}^* I_{2N+1} \mathbf{u}_N \Big| \\
&= \Big| \frac{1}{2N+1} \{ \mathbf{u}_N{}^* [(2N+1)^{-1} U_{NN} U_{NN}^* - I_{2N+1}] \mathbf{u}_N \} \Big| \\
&\leqslant \frac{1}{2N+1} \rho(\mathbf{u}_N{}^*) \rho[(2N+1)^{-1} U_{NN} U_{NN}^* - I_{2N+1}] \rho(\mathbf{u}_N) \\
&\leqslant \frac{1}{2N+1} (2N+1) M \rho[(2N+1)^{-1} U_{NN} U_{NN}^* - I_{2N+1}] M,
\end{aligned}$$

and the last member tends to zero, by condition (IV). (The asterisks on the vectors here indicate merely transposes, since the vectors are real.) We now invoke Lemma 6.3 (b), and Lemma 6.4 follows at once.

Lemma 6.5. *Let $\langle W_N \rangle$ satisfy conditions* (III) *and* (IV). *Then given any $\epsilon > 0$, there exists an $H = H_\epsilon > 0$ such that $\sum_{h=H}^{N} | l_{Nh} |^2 < \epsilon$ for all $N \geqslant H$.*

By referring to the conclusions of Lemma 6.3 and Lemma 6.4, a proof can be given which is identical, step by step, with that provided by Zygmund [21, Vol. 2, p. 15] for the case of trigonometric interpolation in equally spaced point sets W_N.

Lemma 6.6. *Let $\langle W_N \rangle$ satisfy conditions* (III) *and* (IV) *of Section* 4. *Then*

$$\lim_{N\to\infty} \sum_{h=0}^{N} | l_{Nh} - f_h |^2 = 0.$$

This is the result to which Lemmas 6.3–6.5 are addressed. We have

$$\sum_{h=0}^{N} | l_{Nn} - f_h |^2 \leqslant \sum_{h=0}^{H-1} | l_{Nh} - f_h |^2 + \sum_{h=H}^{H} (2 | l_{Nh} |^2 + 2 | f_h |^2).$$

(Here we used the elementary inequality $| z_1 + z_2 |^2 \leqslant 2 | z_1 |^2 + 2 | z_2 |^2$). Given $\epsilon > 0$, first choose H (Lemma 6.5) so that $2 \sum_{h=H}^{N} | l_{Nh} |^2 < \epsilon/3$ for all $N > H$. Parseval's formula for the Fourier series of u implies that $\lim_{M\to\infty} \sum_M^\infty |f_h|^2 = 0$; so we can adjust H upward, if necessary, so that $2\sum_H^N |f_h|^2 < \epsilon/3$. Now hold H fixed. By Lemma 6.3 (a), there exists an $N_0 > 0$ such that $N > N_0$ implies $\sum_{h=0}^{H-1} |l_{Nh} - f_h|^2 < \epsilon/3$. These inequalities together prove the Lemma.

We now return to (6.3). We have ($n \geqslant n_S$)

$$\begin{aligned}\delta_{Nn} &= |g_{N0} - l_{N0} + l_{N0} - f_0|^2 + 2\sum_{h=1}^{n} \Big| g_{Nh} - l_{Nh} + \sum_{j=1}^{n} \bar{\alpha}_{jh}\bar{g}_{Nj} + l_{Nh} - f_h \Big|^2 \\ &\leqslant 2|g_{N0} - l_{N0}|^2 + 4\sum_{h=1}^{n} \Big| g_{Nh} - l_{Nh} + \sum_{j=1}^{n} \bar{\alpha}_{jh}\bar{g}_{Nj}\Big|^2 + 2|l_{N0} - f_0|^2 \\ &\quad + 4\sum_{h=1}^{n} |l_{Nh} - f_h|^2.\end{aligned}$$

By Lemma 6.2, as $N \to \infty$, the first term on the right tends to zero, and as $n \to \infty$, the second term tends to zero, uniformly in $N > n$. By Lemma 6.3 (a), the third term tends to zero as $N \to \infty$. For $n \leqslant N$, the fourth term is dominated by $4\sum_{h=1}^{N} |l_{Nh} - f_h|^2$, and by Lemma 6.6, the latter tends to zero as $N \to \infty$. Thus $\delta_{Nn} \to 0$ as $n \to \infty$, uniformly in $N > n$.

As for ϵ_{Nn}, we have (with $n \geqslant n_S$)

$$\begin{aligned}\epsilon_{Nn} &\leqslant 4\sum_{h=n+1}^{\infty} |f_h|^2 + 4\sum_{h=n+1}^{\infty} \Big|\sum_{j=1}^{n} \bar{\alpha}_{jh}\bar{g}_{Nj}\Big|^2 \\ &\leqslant 4\sum_{h=n+1}^{\infty} |f_h|^2 + 4ng_r^2 \sum_{h=n+1}^{\infty}\sum_{j=1}^{\infty} |\bar{\alpha}_{jh}^2|.\end{aligned}$$

Here we have used the Cauchy inequality and the bound g_r given by Lemma 6.1(b). The first term on the right tends to zero as $n \to \infty$, by Parseval's formula for the Fourier series of u. The second term can be estimated by means of (5.1), and in a routine manner a constant β_r can be exhibited, which depends only on n, such that the double sum in this term is bounded by $\beta_r r^n n^{-1}$. This second term is therefore $O(r^n)$, uniformly in $N > n$.

The proof of Theorem 6.1 is now complete. We return to the proof of Theorem 3.1. As mentioned previously, part (1) of Theorem 3.1 is contained in Lemma 6.1. Part (2) follows from Theorem 6.1 by making the change of variables $z = \phi(w)$, $w = \exp(i\theta)$ in the integral

$$\int_C |H_n(Z_N; u; z) - u(z)|^2 \, |dz|$$

and noting that with C analytic, $|\phi'(w)|$ must be bounded on $|w| = 1$. For part (3) of Theorem 3.1, we use the Poisson integral formula for $H_n - u$ and for Int C. A convenient form of it is obtained by letting $w = \chi(z)$ be any function analytic and univalent on $C \cup \operatorname{Int} C$ which maps this closed region onto $|w| \leqslant 1$; then

$$H_n(Z_N\,; u; z) - U(z)$$
$$= \frac{1}{2\pi} \int_C [H_n(Z_N\,; u, \zeta) - u(\zeta)] \operatorname{Re} \left[\frac{\bar{\chi}(\zeta) + \chi(z)}{\chi(\zeta) - \chi(z)}\right] \chi'(\zeta) \,|\, d\zeta \,|, \quad z \in \operatorname{Int} C.$$

For z on a compact subset of Int C, and ζ on C, $\min |\chi(\zeta) - \chi(z)| > 0$. An application of the Schwarz inequality, together with this fact and part (2) of the theorem establish the almost uniform convergence promised in part (3).

7. Asymptotic Structure of $H_n(Z_N\,; u; z)$

The assumptions stated in the first paragraph of Section 6 are in effect throughout the present section. Lemma 6.5 implies that if $\langle W_N \rangle$ satisfies conditions (III) and (IV), then given any $\epsilon > 0$, for each sufficiently large index h, the Fourier–Lagrange coefficient l_{Nh} satisfies $|l_{Nh}|^2 < \epsilon$ for all $N \geqslant h$. With the aid of Lemma 6.2, a similar result can be obtained for the coefficients of the least-squares harmonic interpolation polynomial.

Theorem 7.1. *Let the sequence of nodes $\langle W_N \rangle$ satisfy conditions* (II)–(IV) *of Section* 4. *Then for any $\epsilon > 0$, there exists an $H = H_\epsilon > 0$ such that $\sum_{h=H}^{N} |g_{Nh}|^2 < \epsilon$ for all $N > n \geqslant H$.*

For the proof, we write

$$g_{Nh} = \left(g_{Nh} - l_{Nh} + \sum_{j=1}^{n} \bar{\alpha}_{jh} \bar{g}_{Nj}\right) + \left(l_{Nh} - \sum_{j=1}^{n} \bar{\alpha}_{jh} \bar{g}_{Nj}\right).$$

The technique of estimation applied in Section 6 to δ_{Nn} and ϵ_{Nn} in (6.3) yields the inequality

$$\sum_{j=H}^{n} |g_{Nh}|^2 \leqslant 2 \sum_{h=H}^{n} \left| g_{Nh} - l_{Nh} + \sum_{j=1}^{n} \bar{\alpha}_{jh} \bar{g}_{Nj} \right|^2$$
$$+ 4 \sum_{h=H}^{\infty} \sum_{j=1}^{\infty} |\alpha_{jh} \bar{g}_{Nj}|^2 + 4 \sum_{h=H}^{N} |l_{Nh}|^2.$$

The proof is completed by applying Lemmas 6.1(b), 6.2(b), 6.5, and the inequalities (5.1) to appropriate terms in the above inequality.

In the remainder of this section we shall assume that W_N consists of the $(2N+1)$th *roots of unity for each N*, so conditions (I–IV) are satisfied automatically. Here $(U_{Nn}^* U_{Nn})^{-1} = (2N+1)^{-1}$. If n is large enough for the normal equations to be unisolvent, we can write (6.1) in the form

$$\mathbf{g}_n^{(N)} = \left(A_n - \frac{1}{2N+1}\mathcal{Q}_n^{(N)}\right)^{-1}\left[\mathbf{l}_n^{(N)} - \frac{1}{2N+1}(A_n{}^*)^{-1}E_{Nn}^*\mathbf{u}_N\right].$$

(We have affixed a superscript to $\mathcal{Q}_n$ to show dependence on N.) The question is: If $n \geqslant n_S$ is held fixed, does the vector $\mathbf{g}_n^{(N)}$ (and therefore the least-squares polynomial $H_n(Z_N\,; u; z)$) tend to some well-defined limiting form as $N \to \infty$?

The answer is in the affirmative, although the formula for the limit is not as simple as one could wish. First, we define a $(2N+1)$-vector $\mathbf{a}_n = (0, a_{n1}, a_{n2}, ..., a_{nn}, b_{n1}, b_{n2}, ..., b_{nn})$, where $a_{nj} = \sum_{h=n+1}^{\infty} \alpha_{jh} f_h$, $j = 1,..., n$, and $b_{nj} = \sum_{h=n+1}^{\infty} \bar{\alpha}_{jh}\bar{f}_h$, $j = 1,..., n$. Here, as in Section 6, the numbers f_h are the Fourier coefficients of $u[\phi(\exp(i\theta))]$. Then, by using Lemma 6.6 and the estimates (5.1), it can be shown that

$$\lim_{N\to\infty} \frac{1}{2N+1} E_{Nn}^* \mathbf{u}_n = \mathbf{a}_n .$$

As usual, let $\mathbf{f}_n = (f_0, f_1, ..., f_n, f_{-1}, ..., f_{-n})$. We know from Lemma 6.3 that $\lim_{N\to\infty} \mathbf{l}_u^{(N)} = \mathbf{f}_n$. It remains to calculate the limit of $(2N+1)^{-1}\mathcal{Q}_n^{(N)} = -U_{Nn}^* E_{Nn} - (A_n{}^*)^{-1}(E_{Nn}^* U_{Nn} A_n + E_{Nn}^* E_{Nn})$. By using the properties (4.1) of the roots of unity, together with the estimate (5.1), we find that $\lim_{N\to\infty} \rho(U_{Nn}^* E_{Nn}) = 0$, $\lim_{N\to\infty} \rho(E_{Nn} U_{Nn}^*) = 0$. We now introduce the partitioned matrix

$$\Gamma_n = \left[\begin{array}{c|c|c} 0 & 0\ \ 0\ \cdots\ 0 & 0\ \cdots\ 0 \\ \begin{matrix}0\\ \vdots \\ 0\end{matrix} & 0_{n+1} & \begin{matrix}\sum_{h=n+1}^{\infty} \alpha_{ih}\bar{\alpha}_{jh} \\ i, j = 1,..., n\end{matrix} \\ \hline \begin{matrix}0\\ \vdots \\ 0\end{matrix} & \begin{matrix}\sum_{h=n+1}^{\infty} \bar{\alpha}_{ih}\alpha_{jh} \\ i, j = 1,..., n\end{matrix} & 0_n \end{array}\right],$$

where O_M is the zero square matrix of order M. By a proper application of the estimates (5.1), it can be shown that $\lim_{N\to\infty}(2N+1)^{-1} E_{Nn}^* E_{Nn} = \Gamma_n$. It follows that

$$\lim_{N\to\infty} \frac{1}{2N+1} \mathfrak{A}_n^{(N)} = -(A_n{}^*)^{-1}\Gamma_n .$$

We put this all together, and obtain

THEOREM 7.2. *Let C be an analytic simple closed curve, let $Z_N = \{\phi(\exp[2\pi k(2N+1)^{-1}]): k = 0, 1,..., 2N\}$, let n be sufficiently large so that the least-squares polynomial $H_n(Z_N; u; z)$, for any continuous $u(z)$ given on C, is uniquely determined, and, finally, let H_n be written as a linear combination of the Faber polynomials $p_1, p_2, \ldots$ belonging to C:*

$$H_n(Z_N ; u; z) = g_{N0} + \sum_{j=1}^{n} [g_{Nj}p_j(z) + \bar{g}_{Nj}\overline{p_j(z)}].$$

Then, with n fixed, $\lim_{N\to\infty} H_n(Z_N; u; z)$ exists, and the vector $\mathbf{g}_n^{(N)} = (g_{N0}, \ldots, g_{Nn}, \bar{g}_{N1}, \ldots, \bar{g}_{Nn})$ has the limiting form

$$\mathbf{g}_n^{(\infty)} = (A_n - (A_n{}^*)^{-1}\Gamma_n)^{-1}(f_n - (A_n{}^*)^{-1}a_n). \tag{7.1}$$

Here A_n is as defined in Section 3, $\mathbf{f}_n$ is the Fourier coefficient vector of $u[\phi(\exp(i\theta))]$ as arranged in Section 6, and Γ_n and $\mathbf{a}_n$ are defined immediately above.

What happens, when $n \to \infty$ in (7.1)? It is clear from the estimates (5.1) that both $\rho(\Gamma_n)$ and $\rho(\mathbf{a}_n)$ are $O(r^n)$. It can be shown that $\lim_{n\to\infty} A_n^{-1}\mathbf{f}_n]_j$ (the limit of the jth component of $A_n^{-1}\mathbf{f}_n$) exists for each fixed j; the details are given in [4, Section 4]. By using these facts and the row norm inequality (5.5) for $(A_n{}^*)^{-1}$, it can be shown that, for each fixed j,

$$\lim_{n\to\infty} g_{nj} = \lim_{n\to\infty} A_n^{-1}\mathbf{f}_n]_j .$$

In loose terms, then, the asymptotic form of $H_n(Z_N; u; z)$ has the coefficient vector $A_\infty^{-1}\mathbf{f}_\infty$. This was established rigorously in [4] for the direct interpolation harmonic polynomials (the case $n = N$, in present notation).

As a final footnote to the general theory, we remark that all the results of Sections 6 and 7 go through without change if $u(z)$ is merely bounded and piecewise continuous on C.

8. Harmonic Polynomial Interpolation on the Unit Circle

In this section we shall examine a case in which C is the unit circle, and the sequence $\langle Z_N \rangle$ is strongly equidistributed on C, and indeed the points of Z_N are equally spaced except for a perturbation. Let $\xi = \exp(i\alpha)$ be a point on the unit circle which is not a root of unity, i.e., $\alpha/2\pi$ is irrational. The Nth set of nodes, Z_N, will be $W_N = \{w_{N0}, ..., w_{N,2N-2}, w_{N,2N-1}, w_{N,2N}\}$, where $w_{Nk} = \exp[2\pi ik/(2N-1)]$, $k = 0,..., 2N-2$, $w_{N,2N-1} = \xi$, $w_{N,2N} = \bar{\xi}$. We shall show that $\lim_{n\to\infty} H_n(W_N; u; z) = U(z)$, uniformly in $N > n$ and almost uniformly on Int C, where H_n is the least-squares polynomial of Section 2 for u and W_N. We shall also show that the sequence $\langle H_N(W_N; u; z)\rangle_{N=1}^{\infty}$ is unbounded for certain choices of u and certain points $z \in$ Int C, where H_N is the harmonic polynomial of degree at most N found by interpolation to u in W_N. The demonstration suggests the possibility of a less erratic convergence behavior for the overdetermined interpolation process than for the simple interpolation process, insofar as spacing of the points on the curve is concerned.

Some preliminary remarks are in order. We shall write H_n in the form (2.2) with $p_j(z) = z^j$; thus $H_n(W_N; u; z) = g_{N0} + \sum_{j=1}^{n} (g_{Nj}z^j + \bar{g}_{Nj}\bar{z}^j)$. The matrix P_{Nn} of Section 2 then reduces to the matrix U_{Nn} of Section 3. As noted in the third paragraph of Section 4, when the points w_{Nk} are distinct, the matrix U_{NN} is nonsingular for $N = 1, 2,...$, and so is $U_{Nn}^* U_{Nn}$ for $n < N$. The question of the existence and uniqueness of $H_n(W_N; u; z)$ for all n and N, $n \leqslant N$, is thus answered affirmatively and requires no further attention. By calculating $(U_{Nn}^* U_{Nn})^{-1}$ explicitly, it is possible to show without too much difficulty that condition (I) of Section 4 is satisfied by $\langle W_N \rangle$, while condition (II) is not. Also, condition (IV) is not satisfied. Condition (III) is clearly satisfied. (These conditions, of course, were merely sufficient conditions for the desired convergence behavior). The sequence $\langle W_N \rangle$ is not "asymptotically neutral" in the sense of Korevaar [11, 16], and therefore cannot be used successfully as a nodal sequence in the parallel problem in the theory of complex polynomial interpolation.

We now introduce the harmonic polynomial $T_{N-1}(u; z)$ of degree at most $N-1$ which interpolates to u in the $(2N-1)$th roots of unity $w_{N0}, ..., w_{N,2N-2}$. Let $l_{Nh} = (2N-1)^{-1} \sum_{k=0}^{2N} \bar{w}_{Nk}^h u(w_{Nk})$, $h = 0, \pm 1, \pm 2, ...$. (The notation is inconsistent with that of Section 3 to the extent that the present l_{Nh} would there be written $l_{N-1,h}$.) Let

$$T_{N-1,n}(u; z) = l_{N0} + \sum_{h=1}^{n} (l_{Nh}z^h + \bar{l}_{Nh}\bar{z}^h)$$

and set $T_{N-1} = T_{N-1,N-1}$.

By using the identity

$$
\begin{aligned}
s_n(z) &= 1 + z + z^2 + \cdots + z^n + \bar{z} + \bar{z}^2 + \cdots + \bar{z}^n \\
&= \begin{cases} 2n+1, & z = 1, \\ (1 - |z|^2)/(|1-z|^2) - 2\operatorname{Re}[z^{n+1}/(1-z)], & z \neq 1, \\ \sin(n+\frac{1}{2})\theta/\sin\theta/2, & z = e^{i\theta}, \end{cases}
\end{aligned} \tag{8.1}
$$

it is easy to derive the formula

$$
\begin{aligned}
T_{N-1,n}(u; z) &= \frac{1}{2N-1} \sum_{k=0}^{2N-2} u(w_{Nk})\, s_n(z\bar{w}_{Nk}) \\
&= \begin{cases} \dfrac{1}{(2N-1)} \displaystyle\sum_{k=0}^{2N-2} u(w_{Nk}) \frac{1 - |z\bar{w}_{Nk}|^2}{|1 - z\bar{w}_{Nk}|^2} & \\ \qquad - \operatorname{Re} \dfrac{2}{2N-1} \displaystyle\sum_{k=0}^{2N-2} u(w_{Nk}) \frac{(z\bar{w}_{Nk})^{n+1}}{1 - z\bar{w}_{Nk}}, & |z| \neq 1, \\ \dfrac{1}{2N-1} \displaystyle\sum_{k=0}^{2N-2} u(w_{Nk}) \frac{\sin(n+\frac{1}{2})(\theta - \theta_{Nk})}{\sin\frac{1}{2}(\theta - \theta_{Nk})}, & \\ \qquad z = e^{i\theta}, \qquad \theta_{Nk} = \operatorname{Arg} w_{Nk}. & \end{cases}
\end{aligned} \tag{8.2}
$$

From the second formula in the third member of (8.2), $T_{N-1}(u; \exp(i\theta))$ is the trigonometric polynomial of degree at most $N - 1$ which interpolates to $u(z) = u[\exp(i\theta)]$ in $2N - 1$ equally spaced points on $[0, 2\pi]$. This interpolation process has been studied extensively by many authors; see Zygmund [21, Chap. 10]. The trigonometric polynomial $T_{N-1,n}(u; \exp(i\theta))$ is called by Zygmund the nth partial sum of T_{N-1}.

When the first sum in the first formula in the third member of (8.2) is multiplied by 2π, it is recognizable as a Riemann sum. Also, if $u(z)$ is bounded on C, the second sum in this formula obviously converges to zero as $n \to \infty$, uniformly in $N > n$ and almost uniformly on Int C. We obtain a result which, in the case $n = N - 1$, is due to Walsh [18]:

THEOREM 8.1. *Let u be continuous on C. Then*

$$
\begin{aligned}
\lim_{n\to\infty} T_{N-1,n}(z) &= \frac{1}{2\pi} \int_0^{2\pi} u(e^{it}) \frac{1 - |z|^2}{|1 - ze^{-it}|^2}\, dt \\
&= \frac{1}{2\pi} \int_0^{2\pi} u(e^{it}) \frac{1 - r^2}{1 - 2r\cos(\theta - t) + r^2}\, dt, \qquad z = re^{i\theta},
\end{aligned}
$$

uniformly in $N > n$ and almost uniformly on Int C.

The integral here is of course the Poisson integral for u, and thus the limit is the solution $U(z)$ of the Dirichlet problem for u and the unit circle.

We proceed now to exhibit an explicit representation for $H_n(W_N\,; u; z) = H_n(z)$, $n < N$. The normal equations $U_{Nn}^{*} U_{Nn} \mathbf{g}_n{}^N = U_{Nn}^{*} \mathbf{u}_N$ can be written in the form

$$g_{N0} + \frac{H_n(\xi) + H_n(\bar{\xi})}{2N-1} = \frac{u(\xi) + u(\bar{\xi})}{2N-1} + l_{N0},$$

$$g_{Nh} + \frac{\xi^{-h} H_n(\xi) + \xi^{h} H_n(\bar{\xi})}{2N-1} = \frac{\bar{\xi}^{h} u(\xi) + \xi^{h} u(\xi)}{2N-1} + l_{Nh},$$

$$\bar{g}_{Nh} + \frac{\xi^{h} H_n(\xi) + \bar{\xi}^{h} H_n(\xi)}{2N-1} = \frac{\xi^{h} u(\xi) + \bar{\xi}^{h} u(\bar{\xi})}{2N-1} + \bar{l}_{Nh},$$

$$h = 1, 2, \ldots, n.$$

By means of some fairly obvious algebraic manipulations, the quantities $H_n(\xi)$, $H_n(\bar{\xi})$ can be eliminated, and we obtain

$$H_n(W_N; u; z) = T_{N-1,n}(z) + R_{N,n}(z),$$

$$R_{N,n}(z) = \frac{[\bar{u}(\xi) - T_{N-1,n}(\xi)][2(n+N)\, s_n(z\bar{\xi}) - s_n(z\xi)\, s_n(\xi^2)]}{4(n+N)^2 - s_n(\xi^2)}$$
$$+ \frac{[u(\bar{\xi}) - T_{N-1,n}(\xi)][2(n+N)\, s_n(z\xi) - s_n(z\bar{\xi})\, s_n(\xi^2)]}{4(n+N)^2 - s_n(\xi^2)},$$

where s_n is given by (8.1).

Certain inequalities for s_n are evident from inspection of (8.1). In the first place, $s_n(\xi^2) \leqslant 2 \mid 1 - \xi^2 \mid^{-1} = A_\xi$. Then, with $\mid z \mid < 1$, $\mid s_n(z\bar{\xi}) \mid \leqslant (1 - \mid z \mid^2)(1 - \mid z \mid)^{-2} + 2(1 - \mid z \mid)^{-1} = (3 + \mid z \mid)(1 - \mid z \mid)^{-1}$, so that given any compact subset D of Int C, there exists a constant $B_D > 0$ such that $\mid s_n(z\bar{\xi}) \mid \leqslant B_D$ for all $z \in D$. Also $\mid s_n(z\xi) \mid \leqslant B_D$ for all $z \in D$. Assuming u is continuous on C, we let $M = \max \mid u(z) \mid$, $z \in C$. By a result due to Faber (see [21, Vol. 2, p. 19]) there exists a sequence $\langle \epsilon_n \rangle_{N=0}^{\infty}$ with $\epsilon_n > 0$, $\epsilon_n \to 0$, such that $\mid T_{N-1,n}[\exp(i\theta)] \mid < \epsilon_n \log n$ for all $N - 1 \geqslant n \geqslant 1$ and all θ. Thus, for $(2(n+N) > A_\xi$, and all $z \in D$, we have

$$\mid R_{N,n}(z) \mid \leqslant \{2(M + \epsilon_n \log n)[2(n+N)\, B_D - B_D A_\xi]\}/[4(n+N)^2 - A_\xi^2]$$

and it is then clear that $R_{N,n}(z) = O(n^{-1} \log n)$, uniformly in $N > n$ and for $z \in D$. We have proved

THEOREM 8.2. *Let u be continuous on C. Then* $\lim_{n\to\infty} H_n(W_N\,; u; z) = U(z)$, *uniformly in $N > n$ and almost uniformly on* Int C.

We now examine the situation for $n = N$. The normal equations for determining the coefficient vector $\mathbf{g}_n^{(N)}$ simplify to $U_{NN}\mathbf{g}_N^{(N)} = \mathbf{u}_N$. The solution is easily obtained, and turns out to be $g_{Nj} = l_{Nj}$, $j = 0, 1, \ldots, N-2$, $g_{N,N-1} = l_{N,N-1} - \bar{g}_{NN}$, and

$$g_{NN} = \frac{[u(\xi) - T_{N-1}(\xi)](\xi^N - \bar{\xi}^{N-1}) - [u(\bar{\xi}) - T_{N-1}(\bar{\xi})](\bar{\xi}^N - \xi^{N-1})}{2i\,\mathrm{Im}[(\xi^N - \bar{\xi}^{N-1})^2]}. \tag{8.3}$$

Substituting these coefficients into $H_N(W_N; u; z)$, we obtain

$$\begin{aligned} H_N(W_N; u; z) &= T_{N-1}(z) + R_N(z), \\ R_N(z) &= (z^N - \bar{z}^{N-1})\, g_{NN} + (\bar{z}^N - z^{N-1})\, \bar{g}_{NN}, \end{aligned} \tag{8.4}$$

with g_{NN} given by (8.3).

For the ensuing divergence analysis, the author is indebted to a suggestion by O'Hara, Jr., to make use of elementary functional analysis. It was pointed out in [1] that we can write H_N in the form $H_N(W_N; u; z) = \sum_{k=0}^{2N} u(w_{Nk})\lambda_{Nk}(z)$, where λ_{Nk} is a harmonic polynomial of degree at most N which vanishes at all points of W_N except w_{Nk}, at which point it has the value unity. It was further remarked that viewed in this light, for a fixed z, say z_0, H_N is a linear functional on the Banach space of continuous functions on C, with the sup norm. The Riesz representation theorem [14, p. 40] permits us to identify the norm of the functional as $\sum_{h=0}^{2N} | \lambda_{Nk}(z_0)|$. Then according to the Banach–Steinhaus Theorem [14, p. 98], if $\sup_N\{\sum_{k=0}^{2N} | \lambda_{Nk}(z_0)|\} = \infty$, it will follow that $\sup_N\{| H_N(W_N; u; z_0)|\} = \infty$ for all u belonging to some dense G_δ-set of functions continuous on C. (This result, in a weaker form, was obtained in [1] by using an ancestor of the Banach–Steinhaus Theorem due to Helly. The curve C in this paragraph can be any simple closed curve, with $W_N \subset C$).

THEOREM 8.3. *Given any point z_0 on the segment* (0, 1) *of the real axis, there exists a complex number ξ, $| \xi | = 1$, ξ not a root of unity, such that the sequence $\langle H_N(W_N; u; z_0)\rangle$ is unbounded for all functions u belonging to some dense G_δ-set of functions continuous on C.*

The plan of the proof, in the light of the preceding paragraph, is to show that $\sup_N | \lambda_{N,2N-1}(z_0)| = \infty$. Now $\lambda_{N,2N-1}(z_0)$ is the coefficient of $u(\xi)$ in $H_N(W_N; u; z_0)$. Since $T_{N-1}(z)$ does not involve $u(\xi)$, this coefficient must lie in $R_N(z)$, and examination of (8.3) and (8.4) reveals it to be

$$\begin{aligned} \lambda_{N,2N-1}(z_0) &= 2\,\mathrm{Re}[(\xi^N - \bar{\xi}^{N-1})(z_0^N - \bar{z}_0^{N-1})/2i\,\mathrm{Im}[(\xi^N - \bar{\xi}^{N-1})^2]] \\ &= -\mathrm{Re}[e^{i(\alpha/2)}(z_0^N - \bar{z}_0^{N-1})/2\sin\alpha(\sin^2(N - \tfrac{1}{2})\alpha)], \end{aligned}$$

where $\xi = \exp(i\alpha)$. Let $z_0 = r$, $0 < r < 1$.

Then

$$\lambda_{N,2N-1}(z_0) = \left(\frac{1}{r} - 1\right) \cos \frac{\alpha}{2} \frac{r^N}{\sin^2(N - \frac{1}{2})\alpha}$$

The proof will be completed by establishing the following result:

LEMMA 8.4. *Given* r, $0 < r < 1$, *and any sequence* $\langle \epsilon(\nu) \rangle_{\nu=1}^{\infty}$ *with* $\epsilon(\nu) > 0$, $\nu = 1, 2,...,$ *and* $\lim_{\nu \to \infty} \epsilon(\nu) = 0$, *there exists an increasing sequence of positive integers* $\langle N_k \rangle$ *and a number* α, $0 < \alpha < 2\pi$, $\alpha/2\pi$ *irrational, such that*

$$| r^{-N_k} \sin(N_k - (1/2))\alpha | < \pi\epsilon(2N_k - 1), \qquad k = 1, 2,... .$$

The Lemma and its proof were suggested by certain results in the doctoral thesis of O'Hara, Jr. [12, Chap. 5].

(In the inequalities (8.5)–(8.7) below, the symbol $[x]$ denotes the largest integer $\leqslant x$; square brackets do not have this meaning elsewhere in this paper.) For any positive integer N,

$$\begin{aligned} \left| \sin \left(N - \frac{1}{2}\right) \alpha \right| &= \left| \sin \left\{ (2N-1) \frac{\alpha}{2\pi} \cdot \pi \right\} - \left[(2N-1) \frac{\alpha}{2\pi} \right] \pi \right\} \right| \\ &\leqslant \pi \left\{ (2N-1) \frac{\alpha}{2\pi} - \left[(2N-1) \frac{\alpha}{2\pi} \right] \right\}. \end{aligned} \tag{8.5}$$

Let $\alpha = 2\pi\beta$, $0 < \beta < 1$, where β is irrational. The number β has a unique infinite simple continued fraction representation; say,

$$\beta = \cfrac{1}{b_1 + \cfrac{1}{b_2 + \cfrac{1}{b_3 + \cdots}}}$$

(see [10, Chap. 10]). Define integers q_n by the recursion $q_0 = 1$, $q_1 = b_1$, $q_n = b_n q_{n-1} + q_{n-2}$, $n \geqslant 2$. Then [10, Theorem 171]

$$q_{2k}\beta - [q_{2k}\beta] < \frac{1}{b_{2k+1} q_{2k}}, \qquad k = 1, 2,... . \tag{8.6}$$

If b_1 is odd, so is q_1. If b_2, b_3,... are even, q_2, q_3,... will be odd. We shall now construct a particular $\beta = \beta_0$ as follows: Take $b_{2k} = 2$, $k = 1, 2,...,$ and take $b_1 = 1$, $b_{2k+1} > (r^{(q_{2k}+1)/2} \epsilon(q_{2k})\, q_{2k})^{-1}$, with b_{2k+1} even. Then q_{2k} is odd, and by (8.5) and (8.6),

$$\begin{aligned} \left| r^{-(q_{2k}+1)/2} \sin \left(q_{2k} \frac{\alpha}{2}\right) \right| &\leqslant r^{-(q_{2k}+1)/2} \cdot \pi(q_{2k}\beta - [q_{2k}\beta]) \\ &\leqslant r^{-(q_{2k}+1)/2} \pi \cdot \frac{r^{(q_{2k}+1)/2} \epsilon(q_{2k})\, q_{2k}}{q_{2k}} = \pi\epsilon(q_{2k}). \end{aligned} \tag{8.7}$$

For the conclusion of the Lemma, we take $\alpha = 2\pi\beta_0$, and $N_k = (q_{2k} + 1)/2$.

It is evident from Theorem 8.1 and the form of $R_N(z)$ in (8.4) that convergence of $\langle H_N \rangle$ does take place at $z = 0$. It is easy to extend Theorem 8.3 to cases in which z_0 is placed in certain locations in Int C other than on the positive real axis (e.g., the negative real axis, the axis of imaginaries), but certain questions remain open. Some of them are: (1) Given *any* $z_0 \in$ Int C, $z_0 \neq 0$, is it always possible to find numbers ξ, $|\xi| = 1$, ξ not a root of unity, such that $\langle H_N(W_N; u; z_0) \rangle$ is unbounded for some continuous function u? (This may be not too difficult to resolve by further study of $\lambda_{N,2N-1}(z_0)$.) Does there exist such a number ξ and a companion continuous function u on C such that $\langle H_N(W_N; u; z_0) \rangle$ diverges for every, or almost all, points z_0 on Int C? Does there exist *any* distribution of point-sets $\langle W_N \rangle$ on C, not necessarily of the special type considered in this section, such that, for some continuous u, the sequence $\langle H_N(W_N; u; z_0) \rangle$ diverges for all, or almost all points $z_0 \in$ Int C?

References

1. J. H. Curtiss, Interpolation with harmonic and complex polynomials to boundary values, *J. of Math. Mech.* **9** (1960), 167–192.
2. J. H. Curtiss, Polynomial interpolation in points equidistributed on the unit circle, *Pac. J. Math.* **12** (1962), 863–877.
3. J. H. Curtiss, Interpolation by harmonic polynomials, *J. Soc. Ind. Appl. Math.* **10** (1962), 709–736.
4. J. H. Curtiss, Solutions of the Dirichlet problem in the plane by approximation with Faber polynomials, *SIAM J. Numer. Anal.*, **3** (1966), 204–228.
5. J. H. Curtiss, Transfinite diameter and harmonic polynomial interpolation, *J. Analyse Math.* **22** (1969), 371–389.
6. J. H. Curtiss, Faber polynomials and the Faber series, *Amer. Math. Monthly*, **78** (1971), 577–596.
7. P. J. Davis, "Interpolation and Approximation," Blaisdell, New York, 1963.
8. L. Fejér, Interpolation und konforme Abbildung, *Nachr. Akad. Wiss. Göttingen. Math.-Phys. Kl. II* (1918), 319–331.
9. H. Grunsky, Koeffizientenbedingungen für schlicht abbildende meromorphe Funktionen, *Math. Z.* **45** (1939), 29–61.
10. G. H. Hardy and E. M. Wright, "An Introduction to the Theory of Numbers," 4th Ed., Clarendon Press, Oxford, 1959.
11. J. Korevaar, Asymptotically neutral distributions of electrons and polynomial approximation, *Ann. of Math.* **80** (1964), 403–410.
12. P. J. O'Hara, "A Study of Interpolation by Complex Polynomials," Ph.D. Dissertation, University of Miami, Coral Gables, Florida, 1968.
13. Ch. Pommerenke, Über die Faberschen Polynome schlichter Funktionen, *Math. Z.* **85** (1964), 197–208.
14. W. Rudin, "Real and Complex Analysis," McGraw–Hill, New York, 1966.
15. O. Taussky, A recurring theorem on determinants, *Amer. Math. Monthly* **56** (1949), 672–675.

16. M. Thompson, Complex interpolation to continuous boundary data, *Proc. Amer. Math. Soc.* **21** (1969), 327–332.
17. J. L. Walsh, On interpolation to harmonic functions by harmonic polynomials, *Proc. Nat. Acad. Sci.*, U.S.A. **18** (1932), 514–517.
18. J. L. Walsh, private communication, 1932.
19. J. L. Walsh, "Interpolation and Approximation by Rational Functions in the Complex Domain," 2nd ed., American Mathematical Society, Providence, RI, 1956.
20. J. L. Walsh, Solution of the Dirichlet problem for the ellipse by interpolating harmonic polynomials, *J. Math. Mech.* **9** (1960), 193–196.
21. A. Zygmund, "Trigonometric series," 2nd ed., Cambridge University Press, London, 1959.